Ma2a Practical – Recitation 8

November 22, 2024

Exercise 1. (Chapter 10.1, Problem 20) Consider the BVP:

$$\begin{cases} x^2y'' - xy' + \lambda y & = & 0 \\ \quad y(1) = 0, & y(L) = 0, & L > 1 \end{cases}$$

Find all the (real) eigenvalues and eigenfunctions of this equation (cf. Theorem 11.2.1). *Hint: it is an Euler equation.*

Exercise 2. Consider Chebyshev's equation (where p is a constant):

$$(1 - x^2)y'' - xy' + p^2y = 0,$$

1. Find two linearly independent series solutions valid for $|x| < 1$ (or at least on an open neighbourhood of 0).
2. *Existence of polynomial solutions.*
   (a) When $p = n$ is an integer, show that there is a polynomial solution to the equation.
   Under the normalization condition $y(1) = 1$, these polynomials are called Chebyshev polynomials and denoted T_p.
   (b) Compute Chebyshev polynomials T_p for $p \in \{0, 1, 2, 3, 4\}$.
3. Back to the general case, solve Chebyshev's equation for $x \in]-1, 1[$ using the change of variable $x = \cos\theta$.

Exercise 3. For Euler equation $x^2y'' - xy' + y = 0$, can you find a series solution $y = \sum_{\geqslant 0} a_n x^n$? Why not?

In fact, 0 is a regular singular point and the formal solution is of a different form. There are exciting developments in ordinary differential equations with regular singularities and irregular singularities. We overview the following in recitation:

1. Regular formal solutions
2. Irregular formal solutions
3. The famous Hilbert 21st problem!

Solution: This is an Euler differential equation, so we look for solutions of the form $y(x) = x^r = e^{r \ln x}$ for $r \in \mathbb{C}$, and determine the possible values of r by plugging back into the equation. We have $y'(x) = rx^{r-1}$ and $y''(x) = r(r-1)x^{r-2}$, so the ODE gives

$$x^2 r(r-1)x^{r-2} - xrx^{r-1} + \lambda x^r = (r(r-1) - r + \lambda)x^r = 0$$

So we want r to satisfy the characteristic equation

$$r^2 - 2r + \lambda = 0$$

The discriminant of this quadratic equation is $\Delta(\lambda) = 4(1-\lambda)$.

For a boundary value problems, solutions typically exhibit an oscillatory behaviour. Solutions will have oscillations if and only if the roots of the characteristic equation are not real, i.e. if and only if $\Delta(\lambda) < 0$. So we expect that all the eigenvalues of the BVP are more than 1. Let's check this.

- <u>If $\lambda > 1$</u>, then the roots are $r_\pm = 1 \pm i\sqrt{\lambda - 1}$. The general solution can be expressed as

  $$y(x) = x\left(c_1 \cos(\ln(x)\sqrt{\lambda-1}) + c_2 \sin(\ln(x)\sqrt{\lambda-1})\right)$$

  Applying the first boundary condition gives

  $$0 = y(1) = c_1 \cos(0) + c_2 \sin(0) = c_1 \Rightarrow c_1 = 0.$$

  Applying the second boundary condition gives us,

  $$0 = y(L) = Lc_2 \sin(\ln(L)\sqrt{\lambda-1})$$

  If $c_2 = 0$ we obtain the trivial solution equal to 0 (not an eigenfunction). If $c_2 \neq 0$, then we

  $$\sin(\ln(L)\sqrt{\lambda-1}) = 0 \Rightarrow \ln(L)\sqrt{\lambda-1} = n\pi, \quad n \in \mathbb{N}_{>0}.$$

  Solving for λ gives us the following set of eigenvalues

  $$\lambda_n = 1 + \frac{n\pi}{\ln L}, \quad n \in \mathbb{N}_{>0}.$$

  The eigenfunctions associated to λ_n are multiples of

  $$y_n(x) = x \sin(\frac{n\pi}{\ln L} \ln(x)).$$

- <u>If $\lambda = 1$</u>, then we get a double root $r = 1$ and the general solution is

  $$y(x) = c_1 x + c_2 x \ln(x).$$

  Applying the first boundary condition gives

  $$0 = y(1) = c_1.$$

The second boundary condition gives

$$0 = y(L) = c_2 L \ln(L) \Rightarrow c_2 = 0.$$

Therefore there are non non-trivial solution to the BVP, and $\lambda = 1$ is not an eigenvalue.

- <u>If $\lambda < 1$</u>, then the roots are $r_{\pm} = 1 \pm \sqrt{1-\lambda}$. So the general solution is

  $$y(x) = c_1 x^{1+\sqrt{1-\lambda}} + c_2 x^{1-\sqrt{1-\lambda}}$$

  Applying the boundary conditions gives

  $$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 L^{1+\sqrt{1-\lambda}} + c_2 L^{1-\sqrt{1-\lambda}} &= 0 \end{aligned}$$

  Because $\lambda \neq 1$, the only solution is $(c_1, c_2) = (0, 0)$. So there are no non-trivial solution to the BVP, and λ is not an eigenvalue.

<u>Conclusion:</u> the eigenvalues and eigenfunctions of this BVP are exactly those described in the case $\lambda > 1$.

Solution

1. We look for a power series solution $y(x) = \sum_{n\in\mathbb{N}} a_n x^n$. Then the differential equation gives the recursive relation

$$\forall n \in \mathbb{N}, \quad a_{n+2} = \frac{n^2 - p^2}{(n+1)(n+2)} a_n \tag{1}$$

The recursion goes by steps of 2, so a solution is uniquely determined by the initial terms (a_0, a_1).

In particular, $(a_0, a_1) = (1, 0)$ and $(a_0, a_1) = (0, 1)$ give two solutions y_1 and y_2. These solutions are linearly independent: if $\lambda y_1 + \mu y_2 = 0$ then evaluating at $x = 0$ gives $\lambda = 0$. Taking the derivative and evaluation at $x = 0$ gives $\mu = 0$.

To check that y_1 and y_2 converge, we can use Theorem 5.3.1 in the textbook: the power series converge in a neighbourhood of 0 where the coefficients $\frac{x}{1-x^2}$ and $\frac{p^2}{1-x^2}$ are analytic (meaning they admit convergent power series expansion at $x = 0$). We have

$$\frac{x}{1-x^2} = \sum_{n\geqslant 0} x^{2n+1} \quad \text{and} \quad \frac{p^2}{1-x^2} = \sum_{n\geqslant 0} p^2 x^{2n}.$$

Those two series converge when $|x| < 1$, so the solutions y_1 and y_2 converge on $(-1, 1)$.

2. (a) The existence of a polynomial solution is equivalent to finding a solution $(a_n)_{n\in\mathbb{N}}$ to the recursion (1) such that $a_n = 0$ for n large enough.

   If p is an even integer, then we claim that the solution associated to te initial conditions $(a_0, a_1) = (1, 0)$ is polynomial. Indeed, since $a_1 = 0$ we have $a_{2n+1} = 0$ for all $n \in \mathbb{N}$. Furthermore $a_{p+2} = 0$, so $a_{p+2n} = 0$ for all $n \in \mathbb{N}_{>0}$.

   If p is an odd integer, then $(a_0, a_1) = (0, 1)$ provides a polynomial solution.

   (b) We look for multiples of the polynomial solution described above.

   - $p = 0$: we have $y_1(x) = 1$, which already satisfies the normalization condition so $T_0(x) = 1$.
   - $p = 1$: we have $y_2(x) = x$ which already satisfies $y_2(1) = 1$, so $T_1(x) = x$.
   - $p = 2$: we have $y_1(x) = -2x^2 + 1$ and $y_1(1) = -1$, so $T_2(x) = \frac{y_1(x)}{y_1(1)} = 2x^2 - 1$.
   - $p = 3$: we have $y_2(x) = -\frac{4}{3}x^3 + x$ and $y_2(1) = \frac{1}{3}$, so $T_3(x) = \frac{y_2(x)}{y_2(1)} = 4x^3 - 3x$.
   - $p = 4$: we have $y_1(x) = 8x^4 - 8x^2 + 1$ and $y_1(1) = 1$, so $T_4(x) = 8x^4 - 8x^2 + 1$.

3. The substitution $x = \cos\theta$ for $0 < \theta < \pi$ is equivalently expressed as $\theta = \arccos x$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{dy}{d\theta} \cdot \frac{-1}{\sqrt{1-x^2}}, \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{d\theta} \cdot \frac{-1}{\sqrt{1-x^2}}\right) = \frac{-x}{(1-x^2)^{\frac{3}{2}}} \cdot \frac{dy}{d\theta} + \frac{1}{x^2-1} \cdot \frac{d^2y}{d\theta^2}
\end{aligned}$$

Since we assume $\theta \in (0, \pi)$ we have $\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$. Plugging these expressions into the ODE we get:

$$-\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} - \frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + p^2y = 0 \iff \frac{d^2y}{d\theta^2} = p^2y.$$

We deduce $y(\theta) = A\cos(p\theta) + B\sin(p\theta)$, and going back to the x variable we see that the general solution is

$$y(x) = A\cos(p\arccos(x)) + B\sin(p\arccos(x)).$$

Remark: the Chebschev polynomial of degree n is usually defined by the equation $T_n(\cos\theta) = \cos(n\theta)$.